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LIGHTLIKE HYPERSURFACES ON A FOUR-DIMENSIONAL MANIFOLD ENDOWED WITH A PSEUDOCONFORMAL STRUCTURE OF SIGNATURE (2, 2)

MAKS A. AKIVIS and VLADISLAV V. GOLDBERG

Dedicated to the memory of Professor Pasquale Calapso

Abstract. *The authors study the geometry of lightlike hypersurfaces on a four-dimensional manifold (M, c) endowed with a pseudoconformal structure $c = CO(2, 2)$. They prove that a lightlike hypersurface $V \subset (M, c)$ bears a foliation formed by conformally invariant isotropic geodesics and two isotropic distributions tangent to these geodesics, and that these two distributions are integrable if and only if V is totally umbilical. The authors also indicate how, using singular points and singular submanifolds of a lightlike hypersurface $V \subset (M, c)$, to construct an invariant normalization of V intrinsically connected with V.*

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0 INTRODUCTION

A four-dimensional pseudo-Riemannian manifold (M, g) with a metric quadratic form of signature $(3, 1)$ is a geometric model of the classic spacetime in general relativity. Its natural generalization is a pseudo-Riemannian manifold (M, g) of dimension $n = \dim\ M$ with a nondegenerate quadratic form of arbitrary signature $(p, q),\ p + q = n$. Such manifolds are considered in a construction of multidimensional models of spacetime and in the theory of superstrings.

Let x be a point of a manifold (M, g), and $T_x(M)$ the tangent space at the point x. For $p \cdot q > 0$, the quadratic form g defines a real *isotropic cone* C_x in the space T_x. Its equation is $g(\xi, \xi) = 0,\ \xi \in T_x$. This cone is also called the *light cone* or the *null cone*.

A hypersurface $V \subset (M, g)$ is called *lightlike* if it is tangent to the cone C_x at each point $x \in V$. The lightlike hypersurfaces are also called *isotropic* or *null*

hypersurfaces. On the manifold (M, g), such hypersurfaces separate domains with different physical or geometric properties—they are models of physical or geometric horizons (see, for example, [HE 73]).

Many physical and geometric objects on a manifold (M, g) are invariant under conformal transformations of the metric g, that is, under a passage from the metric g to the metric $\widetilde{g} = \sigma g$, where $\sigma = \sigma(x)$ is a differentiable function such that $\sigma(x) \neq 0,\ x \in M$. Examples of such objects are the light cones and the lightlike hypersurfaces. Hence it is appropriate to study such objects not only on a pseudo-Riemannian manifold (M, g) but also on a manifold (M, c) endowed with a conformal structure $c = \{\sigma g\}$.

Note that lightlike hypersurfaces arose in the papers of Duggal and Bejancu [DB 91] and [Be 96] (see also their book [DB 96], Section 4.7). They considered them in a pseudo-Riemannian manifold of constant curvature c, and in particular, in pseudo-Euclidean spaces $\mathbf{R}^4_1$ and $\mathbf{R}^4_2$. Lightlike hypersurfaces were also studied by Kupeli [Ku 87] (see also his book [Ku 96], Section 4.4). He considered them in a (pseudo-)Riemannian space (M, g) of constant sectional curvature. Lightlike hypersurfaces appeared in the paper Rosca [Ro 71] in which he studied a pair of lightlike hypersurfaces in 1-to-1 correspondence in a Lorentz manifold.

Note also that the totally umbilical lightlike hypersurfaces in Riemannian and pseudo-Riemannian manifolds (M, g) were extensively studied by many authors. They considered their local and global properties. For example, in the papers [Y 75], [Ak 87], [Ra 87], and [Z 96] the authors found necessary and sufficient conditions for a complete spacelike hypersurface to be totally umbilical in (M, g).

The totally umbilical lightlike hypersurfaces in (M, c) endowed with a conformal or pseudoconformal structure were not yet studied extensively. In the papers [AG 99a] and [AG 99b] we have already studied the geometry of lightlike hypersurfaces V on a manifold (M, c) endowed with a conformal structure c of Lorentzian signature $(n - 1, 1)$.

In the present paper we consider lightlike hypersurfaces on a four-dimensional manifold (M, c) endowed with a conformal structure of ultrahyperbolic signature $(2, 2)$. We find some properties of the structure of such hypersurfaces and prove that they bear two isotropic two-dimensional distributions in addition to the fibration of isotropic geodesics. We also prove that integrability of these distributions is necessary and sufficient for a lightlike hypersurface to be totally umbilical. We constructed an invariant normalization of $V \subset (M, c)$ in a fourth differential neighborhood of a point of V. In the case in question, i.e., for $c = CO(2, 2)$, we were able not only to construct such normalization but also to find a foliation of canonical frames.

For our study of lightlike hypersurfaces on a manifold (M, c), $\dim\ M = 4$, sign $c = (2, 2)$, we use the apparatus developed in [A 96] (see also [AG 96], Ch. 5). As far as we know, the lightlike hypersurfaces on such manifolds are studied in the present paper for the first time.

1 A MANIFOLD (M, c)

Consider a manifold (M, c) endowed with a conformal structure c of signature (p, q), dim $M = n = p + q$. Let x be an arbitrary point of M, $T_x(M)$ be its the tangent space, and $C_x \subset T_x(M)$ be the isotropic cone in $T_x(M)$. The space $T_x(M)$ can be compactified by adding the point at infinity y and the isotropic cone C_y with vertex y. After this enlargement, the space $T_x(M)$ becomes a pseudoconformal space $(C^n_q)_x$ of the same signature (p, q).

Under the Darboux mapping (see [AG 98] or [AG 96], Ch. 1), the space $(C^n_q)_x$ will be mapped onto a hyperquadric $(Q^n_q)_x$ of a projective space P^{n+1}_x of dimension $n + 1$. In the space P^{n+1}_x, the hyperquadric $(Q^n_q)_x$ is defined by the equation $(x, x) = 0$.

We associate a family of projective local frames $\{A_0, A_i, A_{n+1}\}, \; i = 1, \ldots, n$, with this hyperquadric in such a way that $A_0 = x$ and $A_{n+1} = y$, where x and y are points of the hyperquadric $(Q^n_q)_x$ for which $(x, y) \neq 0$. This implies

$$(A_0, A_0) = (A_{n+1}, A_{n+1}) = 0, \;\; (A_0, A_{n+1}) = -1$$

The last condition is obtained by taking an appropriate normalization of the points A_0 and A_{n+1}. Here and in what follows the parentheses denote the scalar product with respect to the quadratic form occurring in the left-hand side of the equation of the hyperquadric $(Q^n_q)_x$.

Denote by T_x and T_y the tangent hyperplanes to $(Q^n_q)_x$ in the points x and y and locate the points A_i at the intersection of these hyperplanes, $A_i \in T_x \cap T_y, \; i = 1, \ldots, n$. Then we find that

$$(A_0, A_i) = (A_{n+1}, A_i) = 0, \;\; (A_i, A_j) = g_{ij},$$

where $\det(g_{ij}) \neq 0, \;$ sign $(g_{ij}) = (p, q)$ (see Figure 1).

Figure 1

Now

$$(A_\xi, A_\eta) = (g_{\xi\eta}) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & g_{ij} & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \xi, \eta = 0, 1, \dots, n+1, \tag{1}$$

and the equation of the hyperquadric $(Q^n_q)_x \subset P^{n+1}_x$ can be written in the form

$$(x, x) = g_{ij} x^i x^j - 2x^0 x^{n+1} = 0.$$

The family of projective frames we have constructed is called a *first order frame bundle associated with the manifold* (M, c).

The isotropic cone C_x is the intersection of the hyperquadric $(Q^n_q)_x$ and the tangent hyperplane T_x:

$$C_x = T_x \cap (Q^n_q)_x,$$

and it is defined by the equation

$$g = g_{ij} \xi^i \xi^j = 0, \ \ \xi = (\xi^i) \in T_x.$$

The group of transformations of the tangent space $T_x(M)$ preserving the invariant cone C_x is the group $G = \mathbf{SO}(p, q) \times \mathbf{H}$, where $\mathbf{SO}(p, q)$ is a special pseudoorthonormal group of signature (p, q) and $\mathbf{H} = \mathbf{R}^+$ is the group of homotheties.

It follows from relations (1) that the equations of infinitesimal displacement in the first-order frame bundle have the form

$$\left\{ \begin{array}{ll} dA_0 = \omega^0_0 A_0 + \omega^i_0 A_i, & \\ dA_i = \omega^0_i A_0 + \omega^j_i A_j & + \omega^{n+1}_i A_{n+1}, \\ dA_{n+1} = \quad\quad \omega^i_{n+1} A_i & - \omega^0_0 A_{n+1}, \end{array} \right. \tag{2}$$

where

$$\begin{array}{l} \omega^i_0 = \omega^i \\ \omega^{n+1}_i = g_{ij}\omega^j, \ \ \omega^i_{n+1} = g^{ij}\omega^0_j, \\ dg_{ij} - g_{ik}\omega^k_j - g_{kj}\omega^k_i = 0, \end{array} \tag{3}$$

and g^{ij} is the inverse tensor of the tensor g_{ij}, i.e., $g^{ik} g_{kj} = \delta^i_j$. Note that the tensor g_{ij} and the 1-forms ω^i are defined in a first-order neighborhood of a point $x \in (M, c)$, the 1-forms ω^0_0 and ω^i_j are defined in its a second-order neighborhood, and the 1-forms ω^0_i are defined in its third-order neighborhood.

The 1-forms ω^i define a displacement of a point A_0 and consequently of a frame $\{A_0, A_i, A_{n+1}\}$ along the manifold M. This is the reason that they are called the *basis forms*. For $\omega^i = 0$, equations (2) take the form

$$\left\{ \begin{array}{ll} \delta A_0 = \pi^0_0 A_0, & \\ \delta A_i = \pi^0_i A_0 + \pi^j_i A_j, & \\ \delta A_{n+1} = \quad\quad \pi^i_{n+1} A_i & - \pi^0_0 A_{n+1}. \end{array} \right. \tag{4}$$

Here δ is the symbol of differentiation for $\omega^i = 0$, i.e., with respect to the fiber parameters of the frame bundle, and $\pi^\xi_\eta = \omega^\xi_\eta(\delta)$. Formulas (4) define admissible transformations in a fiber of a first-order frame bundle. These transformations form the group $G' = G \ltimes \mathbf{T}(n)$ that is obtained by a differential prolongation of the group G acting in the space $T_x(M)$. Here $\mathbf{T}(n)$ is a subgroup of the group G' which is isomorphic to the group of parallel translations, and the symbol $\ltimes$ is the semidirect product (see [AG 96], Ch. 4). Equations (4) show that the group G' is isomorphic to the subgroup of the group of pseudoconformal transformations of the space C^n_q keeping invariant the point $x = A_0$ of this space.

2 THE TENSOR OF CONFORMAL CURVATURE OF A MANIFOLD (M, c), $c = CO(2, 2)$

As was proved in Ch. 4 of the book [AG 96], the structure equations of the manifold (M, c) endowed with a conformal structure of an arbitrary signature (p, q) have the form

$$\left\{\begin{array}{l} d\omega^i = \omega^0_0 \wedge \omega^i + \omega^j \wedge \omega^i_j, \\ d\omega^0_0 = \omega^i \wedge \omega^0_i \\ d\omega^i_j = \omega^0_j \wedge \omega^i + \omega^k_j \wedge \omega^i_k + \omega^{n+1}_j \wedge \omega^i_{n+1} + C^i_{jkl}\omega^k \wedge \omega^l, \\ d\omega^0_i = \omega^0_i \wedge \omega^0_0 + \omega^j_i \wedge \omega^0_j + C_{ijk}\omega^j \wedge \omega^k. \end{array}\right. \tag{5}$$

Here the quantities C^i_{jkl} are defined in a third-order neighborhood of a point $x \in (M, c)$ and form the *tensor of conformal curvature*, also called the *Weyl tensor*. Denote it by the letter C, where $C = (C^i_{jkl})$.

The quantities C_{ijk} are defined in a fourth-order neighborhood of a point $x \in (M, c)$. and for $n \geq 4$, they do not form a tensor. Denote the object C_{ijk} by C', i.e. $C' = (C_{ijk})$. The reason for this notation is that for $n \geq 4$, this object is expressed in terms of the covariant derivatives of the tensor C. For $n \geq 4$, the condition $C = 0$ implies $C' = 0$, and a manifold (M, c) is conformally flat, i.e., it is locally equivalent to a conformal space C^n_q.

For $n = 3$, the tensor $C = (C^i_{jkl})$ is identically equal to 0, and the curvature of the space is defined by the object $C' = (C_{ijk})$ which in this case becomes a tensor. In what follows, we will assume that $n \geq 4$.

The components of the tensor C and the object C' satisfy the equations

$$\left\{\begin{array}{l} C_{ijkl} = g_{im}C^m_{jkl}, \\ C_{ijkl} = -C_{jikl} = -C_{ijlk}, \quad C_{ijkl} = C_{klij}, \\ C_{ijkl} + C_{iklj} + C_{iljk} = 0, \\ C^i_{jki} = 0, \;\; C_{ijk} = -C_{ikj}. \end{array}\right. \tag{6}$$

Note that we will use the notation C not only for the tensor C^i_{jkl} but also for the tensor C_{ijkl}.

3 ISOTROPIC FRAMES FOR $(M, c),\ c = CO(2,2)$

Consider four-dimensional manifold (M, c) endowed with a pseudoconformal structure $c = CO(2,2)$. The fundamental quadratic form of this signature is reduced to the form

$$g = 2(\xi^2\xi^3 - \xi^1\xi^4). \tag{7}$$

It follows that the matrix of its coefficients is

$$(A_i, A_j) = (g_{ij}) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

Note that we changed the signs of components of the tensor g_{ij} in comparison with the book [AG 96] and the paper [A 96].

The isotropic cone $C_x \subset T_x(M)$ is defined by the equation

$$\xi^2\xi^3 - \xi^1\xi^4 = 0.$$

We will clarify a structure of this cone. The last equation can be written in two different ways:

$$\frac{\xi^1}{\xi^3} = \frac{\xi^2}{\xi^4} = -\lambda, \quad \frac{\xi^1}{\xi^2} = \frac{\xi^3}{\xi^4} = -\mu.$$

It follows that the cone C_x carries two families of two-dimensional plane generators. These are defined by the equations

$$\xi^1 + \lambda\xi^3 = 0,\ \ \xi^2 + \lambda\xi^4 = 0 \tag{9}$$

and

$$\xi^1 + \mu\xi^2 = 0,\ \ \xi^3 + \mu\xi^4 = 0. \tag{10}$$

The 2-planes defined by equations (9) are called α-generators, and the 2-planes

defined by equations (10) are called β-generators of the cone C_x.

Figure 2

Under the projectivization of the tangent space $T_x(M)$ with center at a point $x = A_0$, there corresponds a ruled quadric PC_x for the cone C_x where PC_x belongs to a three-dimensional projective space $P_x^3 = PT_x(M)$. With respect to the frame $\{\widetilde{A}_i\}$, where $\widetilde{A}_i = PA_i$, the quadric PC_x is defined by the same equation (7) (see Figure 2).

For the conformal structure $CO(2,2)$, the group of transformations of the tangent space $T_x(M)$ preserving the invariant cone is split into three subgroups: $G = \mathbf{SL}(2) \times \mathbf{SL}(2) \times \mathbf{H}$. The first two of these subgroups transfer the families of α- and β-generators of the cones C_x into themselves and are isomorphic to the group of projective transformations on a projective straight line P^1. As in the general case, the third subgroup is the group of homotheties.

On the manifold (M, c), the isotropic α- and β-generators of the cone C_x form two fiber bundles E_α and E_β with the common base M. The fibers of E_α and E_β are the families of α- and β-generators of the cones C_x. By (9) and (10), these fibers are parameterized by means of nonhomogeneous projective parameters λ and μ and are isomorphic to real projective straight lines $\mathbf{R}P_\alpha$ and $\mathbf{R}P_\beta$. Thus the fiber bundles E_α and E_β can be written as $E_\alpha = (M, \mathbf{R}P_\alpha)$ and $E_\beta = (M, \mathbf{R}P_\beta)$. These fiber bundles are *real twistor fibrations* similar to those introduced on four-dimensional manifolds of Lorentzian signature $(3,1)$ by Penrose (see, for example, [PR 86]).

Consider α- and β-generators of the cone C_x. For $\omega^i = 0$ (i.e., for fixed principal parameters), they are defined by equations (9) and (10). One can easily prove that these generators intersect one another in an isotropic straight line connecting the point $A_0 = x$ with the point

$$B = \lambda\mu A_1 - \lambda A_2 - \mu A_3 + A_4, \tag{11}$$

and that they belong to a three-dimensional subspace of the space $T_x(M)$ defined by the equation

$$\xi^1 + \mu\xi^2 + \lambda\xi^3 + \lambda\mu\xi^4 = 0. \tag{12}$$

This subspace is tangent to the isotropic cone C_x along its generator A_0B and is also called isotropic.

In the space $T_x(M)$, we specialize our moving frame in such a way that its vertex A_1 coincides with the point B and the isotropic straight line A_0B coincides with the straight line A_0A_1. Then the nonhomogeneous projective parameters λ and μ occurring in equations (9) and (10) for isotropic 2-planes $A_0A_1A_2$ and $A_0A_1A_3$ become ∞ , $\lambda = \infty,\ \mu = \infty$, and equations (9) and (10) take the form

$$\xi^3 = 0, \ \ \xi^4 = 0 \tag{13}$$

and

$$\xi^2 = 0, \ \ \xi^4 = 0. \tag{14}$$

The equation of the isotropic subspace (12) containing these isotropic α- and β-generators becomes

$$\xi^4 = 0. \tag{15}$$

4 THE STRUCTURE EQUATIONS OF A MANIFOLD (M, c)

For a manifold (M, c) endowed with a conformal structure $c = CO(2, 2)$, in the isotropic frame bundle, equations (8) and the last equation of (3) imply that

$$\left\{\begin{array}{ll} \omega_1^4 = \omega_2^3 = \omega_3^2 = \omega_4^1 = 0, & \\ \omega_2^4 = \omega_1^3, & \omega_4^2 = \omega_3^1, \\ \omega_3^4 = \omega_1^2, & \omega_4^3 = \omega_2^1, \\ \omega_1^1 + \omega_4^4 = 0, & \omega_2^2 + \omega_3^3 = 0. \end{array}\right. \tag{16}$$

Thus on the manifold (M, c), among the forms ω_i^j only the forms $\omega_1^2,\ \omega_2^1,\ \omega_1^3,\ \omega_3^1,\ \omega_1^1$, and ω_2^2 are independent. Hence on such a manifold (M, c), the structure equations (5) take the form

$$\left\{\begin{array}{l} d\omega^1 = (\omega_0^0 - \omega_1^1) \wedge \omega^1 + \omega^2 \wedge \omega_2^1 + \omega^3 \wedge \omega_3^1, \\ d\omega^2 = (\omega_0^0 - \omega_2^2) \wedge \omega^2 + \omega^1 \wedge \omega_1^2 + \omega^4 \wedge \omega_3^1, \\ d\omega^3 = (\omega_0^0 + \omega_2^2) \wedge \omega^3 + \omega^1 \wedge \omega_1^3 + \omega^4 \wedge \omega_2^1, \\ d\omega^4 = (\omega_0^0 + \omega_1^1) \wedge \omega^4 + \omega^2 \wedge \omega_1^3 + \omega^3 \wedge \omega_1^2, \end{array}\right. \tag{17}$$

$$d\omega_0^0 = \omega^1 \wedge \omega_1^0 + \omega^2 \wedge \omega_2^0 + \omega^3 \wedge \omega_3^0 + \omega^4 \wedge \omega_4^0, \tag{18}$$

$$\left\{\begin{array}{rl} d\omega_1^3 = & \omega_1^0 \wedge \omega^3 + \omega_2^0 \wedge \omega^4 + (\omega_1^1 + \omega_2^2) \wedge \omega_1^3 + \Omega_1^3. \\ d(\omega_1^1 + \omega_2^2) = & \omega_1^0 \wedge \omega^1 + \omega_2^0 \wedge \omega^2 - \omega_3^0 \wedge \omega^3 - \omega_4^0 \wedge \omega^4 \\ & +2\omega_1^3 \wedge \omega_3^1 + \Omega_1^1 + \Omega_2^2, \\ d\omega_3^1 = & \omega_3^0 \wedge \omega^1 + \omega_4^0 \wedge \omega^2 + \omega_3^1 \wedge (\omega_1^1 + \omega_2^2) + \Omega_3^1, \end{array}\right. \tag{19}$$

and

$$
\left\{
\begin{array}{rl}
d\omega_1^2 = & \omega_1^0 \wedge \omega^2 + \omega_3^0 \wedge \omega^4 + (\omega_1^1 - \omega_2^2) \wedge \omega_1^2 + \Omega_1^2, \\
d(\omega_1^1 - \omega_2^2) = & \omega_1^0 \wedge \omega^1 - \omega_2^0 \wedge \omega^2 + \omega_3^0 \wedge \omega^3 - \omega_4^0 \wedge \omega^4 \\
& +2\omega_1^2 \wedge \omega_2^1 + \Omega_1^1 - \Omega_2^2, \\
d\omega_2^1 = & \omega_2^0 \wedge \omega^1 + \omega_4^0 \wedge \omega^3 + \omega_2^1 \wedge (\omega_1^1 - \omega_2^2) + \Omega_2^1.
\end{array}
\right.
\tag{20}
$$

Equations (19) prove that the 1-forms $\omega_1^3,\ \omega_1^1 + \omega_2^2$, and ω_3^1 are fiber forms on the isotropic fiber bundle E_α, and the 2-forms $\Omega_1^3,\ \Omega_1^1 + \Omega_2^2$, and Ω_3^1 are the curvature forms of this fiber bundle. Similarly, it follows from equations (20) that the 1-forms $\omega_1^2,\ \omega_1^1 - \omega_2^2$, and ω_2^1 are fiber forms on the isotropic fiber bundle E_β, and the 2-forms $\Omega_1^2,\ \Omega_1^1 - \Omega_2^2$, and Ω_2^1 are the curvature forms of E_β.

Since each of the indices i, j, k, and l takes only four values 1, 2, 3, 4, equations (6) imply that the tensor of conformal curvature C_{ijkl} has 21 essential nonvanishing components that satisfy 11 independent conditions:

$$
\left\{
\begin{array}{l}
C_{1234} - C_{1324} + C_{1423} = 0, \\
C_{1224} = C_{1334} = C_{1213} = C_{2434} = 0. \\
C_{1314} - C_{1323} = C_{1424} - C_{2324} = 0, \\
C_{1214} + C_{1223} = C_{1434} + C_{2334} = 0, \\
C_{1414} = C_{2323} = C_{1234} + C_{1324}.
\end{array}
\right.
\tag{21}
$$

Hence the tensor C_{ijkl} has 10 independent components in all:

$$
\left\{
\begin{array}{l}
C_{1212} = a_0,\ \ C_{1214} = a_1,\ \ C_{1234} = a_2,\ \ C_{1434} = a_3,\ \ C_{3434} = a_4, \\
C_{1313} = b_0,\ \ C_{1314} = b_1,\ \ C_{1324} = b_2,\ \ C_{1424} = b_3,\ \ C_{2424} = b_4.
\end{array}
\right.
\tag{22}
$$

As a result, the curvature forms of the fiber bundles E_α and E_β can be written as

$$
\begin{array}{rl}
\Omega_1^3 = & -2\ [a_0\omega^1 \wedge \omega^2 + a_1(\omega^1 \wedge \omega^4 - \omega^2 \wedge \omega^3) + a_2\omega^3 \wedge \omega^4], \\
\Omega_1^1 + \Omega_2^2 = & +4\ [a_1\omega^1 \wedge \omega^2 + a_2(\omega^1 \wedge \omega^4 - \omega^2 \wedge \omega^3) + a_3\omega^3 \wedge \omega^4], \\
\Omega_3^1 = & +2\ [a_2\omega^1 \wedge \omega^2 + a_3(\omega^1 \wedge \omega^4 - \omega^2 \wedge \omega^3) + a_4\omega^3 \wedge \omega^4],
\end{array}
\tag{23}
$$

and

$$
\begin{array}{rl}
\Omega_1^2 = & -2\ [b_0\omega^1 \wedge \omega^3 + b_1(\omega^1 \wedge \omega^4 + \omega^2 \wedge \omega^3) + b_2\omega^2 \wedge \omega^4], \\
\Omega_1^1 - \Omega_2^2 = & +4\ [b_1\omega^1 \wedge \omega^2 + b_2(\omega^1 \wedge \omega^4 + \omega^2 \wedge \omega^3) + b_3\omega^2 \wedge \omega^4], \\
\Omega_2^1 = & +2\ [b_2\omega^1 \wedge \omega^3 + b_3(\omega^1 \wedge \omega^4 + \omega^2 \wedge \omega^3) + b_4\omega^2 \wedge \omega^4].
\end{array}
\tag{24}
$$

From equations (23) and (24) it follows the tensor $C = (C_{ijkl})$ of conformal curvature is split into two subtensors A and B, $C = A + B$, where

$$
A = \{a_u\},\ \ B = \{b_u\}, \quad u = 0, 1, 2, 3, 4.
$$

These are the curvature tensors of the fiber bundles E_α and E_β.

If one of subtensors A or B vanishes, then a manifold (M, c) is called *conformally semiflat*. In this case the fiber bundle E_α (respectively, E_β) admits a three-parameter family of two-dimensional integral surfaces V_α (respectively, V_β).

If both subtensors A and B vanish, then the tensor C also vanishes. In this case a manifold (M, c) becomes conformally flat and is locally equivalent to a pseudoconformal space C_2^4. Under the Darboux mapping, a hyperquadric Q_2^4 of a projective space P^5 corresponds for the space C_2^4. Under this mapping, two-dimensional plane generators of the hyperquadric Q_2^4. correspond for two-dimensional integral surfaces V_α and V_β of the fiber bundles E_α and E_β.

5 PRINCIPAL ISOTROPIC BIVECTORS

Suppose that ξ and η are vectors of the space $T_x(M)$, and $p = \xi \wedge \eta$ is a bivector defined by ξ and η. The coordinates of p are

$$p^{ij} = \xi^{[i}\eta^{j]} = \frac{1}{2}(\xi^i\eta^j - \xi^j\eta^i), \;\; p^{ij} = -p^{ji}.$$

The tensor of conformal curvature $C = (C_{ijkl})$ allows us to define the *relative conformal curvature of the bivector* p:

$$C(p) = C_{ijkl}p^{ij}p^{kl}. \tag{25}$$

Let us find relative conformal curvatures of the bivectors p_λ and p_μ defined by α- and β-generators of the isotropic cone C_x of the manifold (M, c). From equations (9) it follows that the vectors ξ_λ and η_λ defining the bivector p_λ are defined by the formulas

$$\xi_\lambda = e_3 - \lambda e_1, \;\; \eta_\lambda = e_4 - \lambda e_2.$$

As a result, the coordinates of the bivector p_λ are

$$p^{12} = \lambda^2, \;\; p^{13} = 0, \;\; p^{14} = -\lambda, \;\; p^{23} = \lambda, \;\; p^{34} = 1, \;\; p^{42} = 0.$$

Substituting these values of coordinates p^{ij} into formula (25) and applying relations (22), we find that

$$\frac{1}{4}C(p_\lambda) = a_0\lambda^4 - 4a_1\lambda^3 + 6a_2\lambda^2 - 4a_3\lambda + a_4.$$

Since the right-hand side of the last equation contains only the components of the curvature tensor A of the isotropic fiber bundle E_α, this formula can be written as

$$\frac{1}{4}A(p_\lambda) = a_0\lambda^4 - 4a_1\lambda^3 + 6a_2\lambda^2 - 4a_3\lambda + a_4. \tag{26}$$

Similarly, the bivector p_μ is defined by the vectors

$$\xi_\mu = e_2 - \mu e_1, \;\; \xi_\mu = e_4 - \mu e_3,$$

and its coordinates are

$$p^{12} = 0, \;\; p^{13} = \mu^2, \;\; p^{14} = -\mu, \;\; p^{23} = -\mu, \;\; p^{34} = 0, \;\; p^{42} = -1.$$

This implies that the relative conformal curvatures of the bivector p_μ is

$$\frac{1}{4} B(p_\mu) = b_0\mu^4 - 4b_1\mu^3 + 6b_2\mu^2 - 4b_3\mu + b_4. \tag{27}$$

The isotropic bivectors whose relative conformal curvature vanishes are called the *principal isotropic bivectors*. By (26) and (27), the values of parameters λ and μ defining such bivectors satisfy the algebraic equations

$$a_0\lambda^4 - 4a_1\lambda^3 + 6a_2\lambda^2 - 4a_3\lambda + a_4 = 0 \tag{28}$$

and

$$b_0\mu^4 - 4b_1\mu^3 + 6b_2\mu^2 - 4b_3\mu + b_4 = 0. \tag{29}$$

Thus in the general case the isotropic cone C_x bears four principal α-generators and the same number of principal β-generators if we count each of these generators as many times as its multiplicity.

On a manifold (M, c), the principal isotropic bivectors form four principal α-distributions and the same number of principal β-distributions. It was proved in [A 96] that *if λ is a multiple root of equation* (28)*, then the principal α-distribution defined by this root is integrable.* In the same way *if μ is a multiple root of equation* (29)*, then the principal β-distribution defined by this root is integrable.*

Suppose that λ and μ are two fixed roots of equations (28) and (29), respectively, and p_λ and p_μ are the principal isotropic distributions defined by these two roots. By means of a frame transformation indicated at the end of Section **3**, the values of parameters λ and μ can be made to equal ∞, $\lambda = \infty, \; \mu = \infty$. As a result, by (9) and (10), we find that these two distributions are defined by the following two systems of equations:

$$\omega^3 = 0, \;\; \omega^4 = 0 \tag{30}$$

and

$$\omega^2 = 0, \;\; \omega^4 = 0. \tag{31}$$

Moreover, equations (28) and (29) become

$$-4a_1\lambda^3 + 6a_2\lambda^2 - 4a_3\lambda + a_4 = 0 \tag{32}$$

and

$$-4b_1\mu^3 + 6b_2\mu^2 - 4b_3\mu + b_4 = 0. \tag{33}$$

If the coefficient a_1 in equation (32) vanishes, then the root $\lambda = \infty$ of this equation is a multiple root, and as a result, the principal distribution (30) defined by this root is integrable. Two-dimensional integral surfaces V_α of this distribution form an isotropic fiber bundle on the manifold (M, c). Similarly, if the coefficient b_1 in equation (33) vanishes, then the root $\mu = \infty$ of this equation is a multiple root and the principal distribution (31) defined by this root is integrable. In addition, the two-dimensional integral surfaces V_β of this distribution form an isotropic fiber bundle on the manifold (M, c).

6 LIGHTLIKE HYPERSURFACES ON (M, c), $c = CO(2, 2)$

As we already said in the introduction, a hypersurface V on a manifold (M, c), $\dim V = 3$, is said to be *lightlike* if its tangent subspace $T_x(V)$ is *tangent to the isotropic cone* C_x, i.e., this subspace is isotropic. The aim of this paper is to study the geometry of lightlike hypersurfaces on a manifold (M, c), where $c = CO(2, 2)$.

With a point x of a lightlike hypersurface V, we associate a moving frame in such a way that its vertex A_0 coincide with $x \in V$, $A_0 = x$, the points A_1, A_2, and A_3 belong to the tangent subspace $T_x(V)$, and the point A_1 belongs to the isotropic straight line along which the subspace $T_x(V)$ is tangent to the isotropic cone C_x. The subspace $T_x(V)$ contains two isotropic α- and β-planes intersecting one another along the straight line A_0A_1. Thus the 2-plane $A_0 \wedge A_1 \wedge A_2$ is an α-generator of the cone C_x, and the 2-plane $A_0 \wedge A_1 \wedge A_3$ is its β-generator.

We place points A_2 and A_3 of our moving frame to these two planes and normalize them by the condition $(A_2, A_3) = 1$. The subspace $A_0 \wedge A_2 \wedge A_3$ is called the *screen subspace* and is denoted by S_x , $S_x = A_0 \wedge A_2 \wedge A_3 \subset T_x(V)$. Further we take a point A_4 on the isotropic cone C_x in such a way that the subspace $A_0 \wedge A_1 \wedge A_4$ is conjugate to the subspace S_x with respect to the cone C_x. In addition, we normalize the points A_1 and A_4 by the condition $(A_1, A_4) = -1$.

A straight line $N_x = A_0 \wedge A_4$ does not belong to the tangent subspace $T_x(V)$. This line is called a *normalizing straight line*. Its location is uniquely determined by the location of the subspace S_x.

The matrix of scalar products of the points A_i. $i = 1, 2, 3, 4$, now has the form (8).

the family of frames we have constructed is called a *family of first-order frames associated with a point x of a lightlike hypersurface $V \subset (M, c)$.*

We will now find the equations of a bundle of first-order isotropic frames associated with a lightlike hypersurface V. Since its tangent subspace $T_x(V) = A_0 \wedge A_1 \wedge A_2 \wedge A_3$, with respect to this frame bundle the equation of V is

$$\omega^4 = 0, \tag{34}$$

and as a result, we have

$$dA_0 = \omega_0^0 A_0 + \omega^1 A_1 + \omega^2 A_2 + \omega^3 A_3. \tag{35}$$

The 1-forms ω^1, ω^2, and ω^3 are independent. They are *basis forms* of the frame bundle in question and of the hypersurface V.

Equations

$$\omega^2 = \omega^3 = 0 \tag{36}$$

define on V a foliation formed by isotropic lines. As was proved in [AG 99b], these lines are isotropic geodesics for all pseudo-Riemannian metrics g compatible with the conformal structure $c = CO(2,2)$ on the manifold (M, c).

We will assume that the isotropic geodesics defined by equations (36) can be prolonged indefinitely on a hypersurface V. In this case each of these geodesics bears the geometry of a projective straight line P^1, and a hypersurface V is the image of the product $M^2 \times P^1$ under its differentiable mapping f into the manifold (M, c): $V = f(M^2 \times P^1),\ f : M^2 \times P^1 \to (M, c)$.

Equation $\omega^3 = 0$ defines on V a fibration of isotropic α-planes $A_0 \wedge A_1 \wedge A_2$, and equation $\omega^2 = 0$ defines on V a fibration of isotropic β-planes $A_0 \wedge A_1 \wedge A_3$ (cf. these two equations with equations (30) and (31)).

In an isotropic frame bundle, the first fundamental form I of a lightlike hypersurface $V \subset (M, c)$ becomes

$$I = g|_V = (dA_0, dA_0) = 2\omega^2\omega^3. \tag{37}$$

This form is of rank 2 and of signature $(1,1)$, and its coefficients form the matrix

$$(g_{ab}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ a, b = 2, 3. \tag{38}$$

7 SINGULAR POINTS AND TOTALLY UMBILICAL HYPERSURFACES

By the last equation of (17), exterior differentiation of equation (34) (the basic equation of a lightlike hypersurface V) leads to the following exterior quadratic equation:

$$\omega^2 \wedge \omega_1^3 + \omega^3 \wedge \omega_1^2 = 0. \tag{39}$$

Applying Cartan's lemma to this equation, we that

$$\left\{ \begin{array}{l} \omega_1^3 = \lambda_{22}\omega^2 + \lambda_{23}\omega^3, \\ \omega_1^2 = \lambda_{32}\omega^2 + \lambda_{33}\omega^3, \end{array} \right. \tag{40}$$

where $\lambda_{23} = \lambda_{32}$.

By means of the Cartan test (see [BCGGG 91] and cf. [AG 99a]), one can prove that lightlike hypersurfaces $V \subset (M, c)$, where $c = CO(2,2)$, exist and depend on a function of two variables.

Differentiating equation (35), we obtain

$$d^2 A_0 \equiv (\omega^2 \omega_2^4 + \omega^3 \omega_3^4) A_4 + (\omega^2 \omega_2^5 + \omega^3 \omega_3^5) A_5 \pmod{T_x(V)}. \tag{41}$$

But by (3) and (8) we have

$$\omega_2^5 = \omega^3, \;\; \omega_3^5 = \omega^2, \;\; \omega_2^4 = \omega_1^3, \;\; \omega_3^4 = \omega_1^2.$$

Thus by (40) relation (41) takes the form

$$d^2 A_0 \equiv (\lambda_{22} (\omega^2)^2 + 2\lambda_{23} \omega^2 \omega^3 + \lambda_{33} (\omega^3)^2) A_4 + 2\omega^2 \omega^3 A_5 \pmod{T_x(V)}. \tag{42}$$

Note that the coefficient in A_5 in equation (42) coincides with the first fundamental form (37) of a hypersurface $V \subset (M, c)$.

Denote by $\widetilde{II}$ the coefficient in A_4 in equation (42):

$$\widetilde{II} = \lambda_{22} (\omega^2)^2 + 2\lambda_{23} \omega^2 \omega^3 + \lambda_{33} (\omega^3)^2.$$

Then equation (42) takes the form

$$d^2 A_0 = \widetilde{II} A_4 + I A_5 \pmod{T_x(V)}. \tag{43}$$

If we multiply expression (43) by a point $A_1 - x A_0$, then by (1) and (8), we find that

$$(d^2 A_0, A_1 - x A_0) = -(\widetilde{II} - x I).$$

The expression in the parentheses of the right-hand side is a *pencil* of the second fundamental forms of a hypersurface $V \subset (M, c)$:

$$\widetilde{II} - x I = \lambda_{22} (\omega^2)^2 + 2(\lambda_{23} - x) \omega^2 \omega^3 + \lambda_{33} (\omega^3)^2. \tag{44}$$

The matrix of their coefficients is

$$(\widetilde{\lambda}_{ab}) = \begin{pmatrix} \lambda_{22} & \lambda_{23} - x \\ \lambda_{23} - x & \lambda_{33} \end{pmatrix}.$$

From the pencil (43) we will take the form whose matrix is apolar to the matrix (g_{ab}), that is, the matrix satisfying the condition

$$\widetilde{\lambda}_{ab} g^{ab} = 0. \tag{45}$$

Since by (38) we have

$$(g^{ab}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

it follows that

$$\widetilde{\lambda}_{ab} g^{ab} = 2(\lambda_{23} - x).$$

Thus condition (45) leads to the relation

$$x = \lambda_{23}. \tag{46}$$

Condition (45) singles out from a pencil (44) of the second fundamental forms of a hypersurface $V \subset (M, c)$ a conformally invariant fundamental form

$$II = \widetilde{II} - \lambda_{23} I = \lambda_{22}(\omega^2)^2 + \lambda_{33}(\omega^3)^2. \tag{47}$$

Its matrix has the form

$$(h_{ab}) = (\lambda_{ab}) - \lambda_{23}(g_{ab}) = \begin{pmatrix} \lambda_{22} & 0 \\ 0 & \lambda_{33} \end{pmatrix} \tag{48}$$

and is diagonal.

Consider singular points of the map $f(M^2 \times P^1) = V^3 \subset (M, c)$. We will look for these points in the form $X = A_1 - sA_0$. At these points the dimension of the tangent subspace $T_X(V)$ must be reduced. By (3), (11), and (34), we have

$$dA_1 = \omega_1^0 A_0 + \omega_1^1 A_1 + \omega_1^2 A_2 + \omega_1^3 A_3. \tag{49}$$

Applying equations (49) and (35), we find that

$$d(A_1 - sA_0) = (\omega_1^0 - x\omega_0^0 - dx)A_0 + (\omega_1^1 - x\omega_0^1)A_1 + (\omega_1^2 - x\omega_0^2)A_2 + (\omega_1^3 - x\omega_0^3)A_3.$$

Further by (40) we obtain

$$d(A_1 - sA_0) \equiv ((\lambda_{23} - s)A_2 + \lambda_{22}A_3)\omega^2 + (\lambda_{33}A_2 + (\lambda_{23} - s)A_3\omega^3)(\pmod A)_0, A_1).$$

The tangent subspace $T_X(V)$ is determined by the points $A_0, A_1, (\lambda_{23} - s)A_2 + \lambda_{22}A_3$, and $\lambda_{33}A_2 + (\lambda_{23} - s)A_3$. Thus the dimension of the tangent subspace is reduced only in the points $X = A_1 - sA_0$ in which

$$\det \begin{pmatrix} \lambda_{23} - s & \lambda_{22} \\ \lambda_{33} & \lambda_{23} - s \end{pmatrix} = 0.$$

This equation can be written as

$$s^2 - 2\lambda_{23}s + (\lambda_{23}^2 - \lambda_{22}\lambda_{33}) = 0. \tag{50}$$

Denote by s_1 and s_2 the roots of this equation. They are calculated by the following formula:

$$s_{1,2} = \lambda_{23} \pm \sqrt{\lambda_{22}\lambda_{33}}.$$

The points $F_1 = A_1 - s_1 A_0$ and $F_2 = A_1 - s_2 A_0$ are singular points of an isotropic geodesic $l = A_0 A_1$ of a hypersurface V.

By Vieta's theorem, it follows from equation (50) that

$$s_1 + s_2 = 2\lambda_{23}.$$

Thus the point $H = A_1 - \lambda_{23} A_0$ is the fourth harmonic point H of the point A_0 with respect to the points F_1 and F_2 on the line $l = A_0 A_1$. The singular points F_1 and F_2 are located symmetrically with respect to the points A_0 and H.

Now the conformally invariant second fundamental form II of a hypersurface $V \subset (M, c)$ can be written as

$$II = -(d^2 A_0, H).$$

We take a moving frame whose vertex A_1 coincides with the point H. This implies $\lambda_{23} = 0$. As a result, equation (50) becomes

$$s^2 - h_{22} h_{33} = 0,$$

and

$$s_{1,2} = \pm\sqrt{h_{22} h_{33}}. \tag{51}$$

The following theorem follows from relation (51).

Theorem 1 **(a)** *The second fundamental form II of a hypersurface $V \subset (M, c)$ at a point A_0 is positive definite or negative definite if and only if the isotropic geodesic $l = A_0 A_1$ through the point $x = A_0$ bears two real singular points. If at a point $x = A_0$ this form is an indeterminate form of rank two, then the singular points on the straight line $l = A_0 A_1$ are complex conjugate.*

(b) *The second fundamental form II of a hypersurface $V \subset (M, c)$ at a point $x = A_0$ has the rank less than two if and only if the singular points on the isotropic geodesic $l = A_0 A_1$ coincide. In this case the point H coincides with this multiple singular point.*

On a lightlike hypersurface V, 2-planes $A_0 \wedge A_1 \wedge A_2$ and $A_0 \wedge A_1 \wedge A_3$ of an isotropic frame bundle compose an α- and β-distribution. Denote them by Δ_α and Δ_β. These distributions are defined on V by the equations

$$\omega^3 = 0 \qquad (\alpha) \qquad \omega^2 = 0. \qquad (\beta) \tag{52}$$

In general, the distributions Δ_α and Δ_β are not integrable. Let us find the conditions of their integrability.

Exterior differentiation of equation (52α) gives the following exterior quadratic equation

$$\omega^1 \wedge \omega_1^3 = 0.$$

Substituting the value of the form ω_1^3 from (40) into this equation and taking into account (48), we find that the distribution Δ_α is integrable if and only if

$$h_{22} = 0. \tag{53}$$

Similarly the distribution Δ_β is integrable if and only if

$$h_{33} = 0. \tag{54}$$

Comparing the conditions (53) and (54) with relations (51) we arrive at the following result.

Theorem 2 *If at least one of the isotropic distributions Δ_α and Δ_β on a lightlike hypersurface $V \subset (M, c)$ is integrable, then the singular points on each of its isotropic generators coincide.*

If both isotropic distributions Δ_α and Δ_β are integrable on a hypersurface V, then conditions (53) and (54) are satisfied simultaneously, and the second fundamental form II of V vanishes. But this means that hypersurface V is totally umbilical. This implies the following result.

Theorem 3 *Both isotropic distributions Δ_α and Δ_β on a lightlike hypersurface $V \subset (M, c)$ are integrable if and only if the hypersurface V is totally umbilical.*

8 SOME PROPERTIES OF LIGHTLIKE HYPERSURFACES

We will pass now to the study of properties of a lightlike hypersurface $V \subset (M, c)$ connected with third- and higher-order differential neighborhoods.

We make a reduction in our isotropic second-order frame bundle by taking a specialized frame whose vertex $A_1 \in l$ coincides with the fourth harmonic point H of the point A_0 with respect to the singular points F_1 and F_2 of the straight line $l = A_0 A_1$. Then we obtain

$$\lambda_{23} = 0, \;\; h_{22} = \lambda_{22}, \;\; h_{33} = \lambda_{33},$$

and equations (40) become

$$\omega_1^3 = h_{22}\omega^2, \;\; \omega_1^2 = h_{33}\omega^3. \tag{55}$$

By (12), (14), (15), (18), and (19), exterior differentiation of equations (55) gives

$$\left\{ \begin{array}{l} \Delta h_{22} \wedge \omega^2 + (-\omega_1^0 + h_{22}h_{33}\omega^1 - 2a_1\omega^2 - 2b_1\omega^3) \wedge \omega^3 = 0, \\ (-\omega_1^0 + h_{22}h_{33}\omega^1 - 2a_1\omega^2 - 2b_1\omega^3) \wedge \omega^2 + \Delta h_{33} \wedge \omega^3 = 0, \end{array} \right. \tag{56}$$

where

$$\left\{ \begin{array}{l} \Delta h_{22} = dh_{22} + h_{22}(\omega_0^0 - 2\omega_2^2 - \omega_1^1) + 2a_0\omega^1, \\ \Delta h_{33} = dh_{33} + h_{33}(\omega_0^0 + 2\omega_2^2 - \omega_1^1) + 2b_0\omega^1. \end{array} \right. \tag{57}$$

By Cartan's lemma, it follows from (56) that

$$\begin{cases} \Delta h_{22} = h_{222}\omega^2 + h_{223}\omega^3, \\ \omega_1^0 = h_{22}h_{33}\omega^1 - (h_{223} + 2a_1)\omega^2 - (h_{233} + 2b_1)\omega^3, \\ \Delta h_{33} = h_{233}\omega^2 + h_{333}\omega^3. \end{cases} \tag{58}$$

We will apply now equations (58) to totally umbilical hypersurfaces $V \subset (M, c)$. For such hypersurfaces we have $h_{22} = h_{33} = 0$. As a result, equations (55) take the form

$$\omega_1^3 = 0, \;\; \omega_1^2 = 0, \tag{59}$$

and equations (58) imply that

$$a_0 = 0, \;\; b_0 = 0, \tag{60}$$

$$h_{222} = h_{223} = h_{233} = h_{333} = 0, \tag{61}$$

and

$$\omega_1^0 = -2(a_1\omega^2 + b_1\omega^3). \tag{62}$$

Conditions (60) mean that the isotropic distributions Δ_α and Δ_β are principal. Moreover, it follows now from equations (49) that

$$dH = \omega_0^0 H - 2(a_1\omega^2 + b_1\omega^3)A_0. \tag{63}$$

This implies the following result.

Theorem 4 *A lightlike totally umbilical hypersurface $V \subset (M, c)$ possesses the following properties:*

(a) *The isotropic distributions Δ_α and Δ_β are integrable and principal.*

(b) *The multiple singular point H of the isotropic geodesic $l = A_0A_1$ describes an isotropic line tangent to the straight line l at the point H.*

(c) *If $a_1 = b_1 = 0$, then the point H is fixed, and a totally umbilical hypersurface is an isotropic cone C_H with vertex H.*

Proof. The statement (a) follows from the fact that on a hypersurface V, the isotropic distributions Δ_α and Δ_β are defined by equations (52) and correspond to the values $\lambda = \infty$ and $\mu = \infty$ in equations (9) and (10). Hence for $a_0 = b_0 = 0$, these values satisfy equations (32) and (33) defining the principal isotropic distributions.

The statement (b) follows immediately from equation (63).

Note that the conditions $a_1 = b_1 = 0$ along with conditions (60) imply that the values $\lambda = \infty$ and $\mu = \infty$ are multiple roots of equations (32) and (33). This implies that the statement (c) can be also formulated as follows:

(c') A lightlike totally umbilical hypersurface $V \subset (M, c)$ is an isotropic cone if and only if it bears multiple isotropic distributions Δ_α and Δ_β.

Note also that in this case the integral surfaces of the distributions Δ_α and Δ_β on a hypersurface V are two-dimensional plane generators of the cone C_H.

9 CONSTRUCTION OF A CANONICAL DISTRIBUTION OF FRAMES FOR A LIGHTLIKE HYPERSURFACE

We associated a family of the second-order frames with a point $x = A_0$ of a lightlike totally umbilical hypersurface $V \subset (M, c)$ in such away that the vertex A_1 coincides with the harmonic pole H of the isotropic tangent A_0A_1. But the points A_2 and A_3 of these frames can move freely in α- and β-planes containing the straight line A_0A_4, and its point A_4 can move freely along the isotropic straight line A_0A_1 that is conjugate to the screen subspace $A_0 \wedge A_2 \wedge A_3$ with respect to the isotropic cone C_x.

For a fixed point $x = A_0$, by equations (16) and (40), we find that

$$\begin{aligned} \delta A_2 &= \pi_2^0 A_0 + \pi_2^1 A_1 + \pi_2^2 A_2, \\ \delta A_3 &= \pi_3^0 A_0 + \pi_3^1 A_1 + \pi_3^3 A_3. \end{aligned}$$

Here $\pi_2^0, \pi_2^1, \pi_3^0$, and π_3^1 are fiber forms defining a displacement of the points A_2 and A_3 in the corresponding isotropic 2-planes.

In order to find the points A_2 and A_3 uniquely in these 2-planes, one needs to make the above mentioned fiber forms vanish. However, this must be done in such a way that a fixing of the points A_2 and A_3 would be intrinsically connected with the geometry of a hypersurface V. The latter can be achieved by fixing in a certain way the coefficients h_{abc} occurring in equations (58). These coefficients are associated with a third-order neighborhood of a hypersurface V.

To this end, we take exterior derivatives of equations (58). As a result, we obtain the following exterior quadratic equations:

$$\left\{ \begin{array}{l} \Delta h_{222} \wedge \omega^2 + \Delta h_{223} \wedge \omega^3 + H_{22} = 0, \\ \Delta h_{223} \wedge \omega^2 + \Delta h_{233} \wedge \omega^3 + H_{23} = 0, \\ \Delta h_{233} \wedge \omega^2 + \Delta h_{333} \wedge \omega^3 + H_{33} = 0, \end{array} \right. \tag{64}$$

where

$$\begin{aligned} \Delta h_{222} &= dh_{222} + h_{222}(2\omega_0^0 - 3\omega_2^2 - \omega_1^1) + 2a_0\omega_2^1 + 3h_{22}\omega_2^0 - 3(h_{22})^2\omega_3^1, \\ \Delta h_{223} &= dh_{223} + h_{223}(2\omega_0^0 - \omega_2^2 - \omega_1^1) + 2a_0\omega_3^1 - h_{22}\omega_3^0 + h_{22}h_{33}\omega_2^1, \\ \Delta h_{233} &= dh_{233} + h_{233}(2\omega_0^0 + \omega_2^2 - \omega_1^1) + 2b_0\omega_2^1 - h_{33}\omega_2^0 + h_{22}h_{33}\omega_3^1, \\ \Delta h_{333} &= dh_{333} + h_{333}(2\omega_0^0 + 3\omega_2^2 - \omega_1^1) + 2b_0\omega_3^1 + 3h_{33}\omega_3^0 - 3(h_{33})^2\omega_2^1. \end{aligned}$$

and H_{22}, H_{23}, and H_{33} are 2-forms that are linear combinations of the products $\omega^1 \wedge \omega^2$, $\omega^2 \wedge \omega^3$, and $\omega^1 \wedge \omega^3$ of the basis forms ω^1, ω^2, and ω^3.

Equations (64) imply that the 1-forms Δh_{222}, Δh_{223}, Δh_{233}, and Δh_{333} are linear combinations of the basis forms ω^1, ω^2, and ω^3.

For a fixed point $x = A_0$, i.e., for $\omega^1 = \omega^2 = \omega^3 = 0$, these forms vanish, and their expressions become

$$\begin{aligned}
\Delta_\delta h_{222} &= \delta h_{222} + h_{222}(2\pi_0^0 - 3\pi_2^2 - \pi_1^1) + 2a_0\pi_2^1 + 3h_{22}\pi_2^0 - 3(h_{22})^2\pi_3^1 = 0,\\
\Delta_\delta h_{223} &= \delta h_{223} + h_{223}(2\pi_0^0 - \pi_2^2 - \pi_1^1) + 2a_0\pi_3^1 - h_{22}\pi_3^0 + h_{22}h_{33}\pi_2^1 = 0,\\
\Delta_\delta h_{233} &= \delta h_{233} + h_{233}(2\pi_0^0 + \pi_2^2 - \pi_1^1) + 2b_0\pi_2^1 - h_{33}\pi_2^0 + h_{22}h_{33}\pi_3^1 = 0,\\
\Delta_\delta h_{333} &= \delta h_{333} + h_{333}(2\pi_0^0 + 3\pi_2^2 - \pi_1^1) + 2b_0\pi_3^1 + 3h_{33}\pi_3^0 - 3(h_{33})^2\pi_2^1 = 0.
\end{aligned} \tag{65}$$

Equations (65) contain the fiber forms π_2^0, π_2^1, π_3^0, and π_3^1 defining a displacement of the points A_2 and A_3 in the α- and β-planes $A_0 \wedge A_1 \wedge A_2$ and $A_0 \wedge A_1 \wedge A_3$. Consider the determinant D of the matrix of coefficients in these fiber forms in equations (65):

$$D = \det \begin{pmatrix} 3h_{22} & 2a_0 & 0 & -3(h_{22})^2 \\ 0 & h_{22}h_{33} & -h_{22} & 2a_0 \\ -h_{33} & 2b_0 & 0 & 3h_{22}h_{33} \\ 0 & -3(h_{33})^2 & 3h_{33} & 2b_0 \end{pmatrix}.$$

Calculating this determinant, we find that

$$D = 4(3h_{22}b_0 + h_{33}a_0)(h_{22}b_0 + 3h_{33}a_0). \tag{66}$$

If this determinant does not vanish, $D \neq 0$, then equations (65) imply that the quantities h_{222}, h_{223}, h_{233}, and h_{333}, occurring in equations (58) can be simultaneously reduced to 0 by means of the fiber forms $\pi_2^0, \pi_2^1, \pi_3^0$, and π_3^1 (see [O 62]). As a result, the points A_2 and A_3 are uniquely determined in the planes $\alpha = A_0 \wedge A_1 \wedge A_2$ and $\beta = A_0 \wedge A_1 \wedge A_3$, and we arrive at a family of third-order moving frames associated with a point $x = A_0 \in V \subset (M, c)$.

With respect to a third-order frame we have constructed, equations (58) take the form

$$\left\{ \begin{array}{l} dh_{22} + h_{22}(\omega_0^0 - 2\omega_2^2 - \omega_1^1) = -2a_0\omega^1, \\ \omega_1^0 = h_{22}h_{33}\omega^1 - 2a_1\omega^2 - 2b_1\omega^3, \\ dh_{33} + h_{33}(\omega_0^0 + 2\omega_2^2 - \omega_1^1) = -2b_0\omega^1. \end{array} \right. \tag{67}$$

This proves the following result.

Theorem 5 *If on a lightlike hypersurface V the determinant D does not vanish, then it is possible to construct a third-order frame bundle on V intrinsically connected with the geometry of V. In this frame bundle, $h_{abc} = 0$.*

Note that if the $CO(2,2)$-structure on a manifold (M,c) is conformally flat, then a third-order frame bundle indicated above cannot be constructed since for a conformally flat structure we have $a_0 = b_0 = 0$, and consequently, $D = 0$. However, for a conformally semiflat $CO(2,2)$-structure the above construction is possible. A construction of a canonical frame bundle for lightlike totally umbilical hypersurfaces is also impossible since for them $h_{22} = h_{33} = 0$, and consequently, $D = 0$.

In order to complete our construction of a canonical frame bundle, we also have to fix the vertex A_4 on the isotropic straight line A_0A_4 which is conjugate to the screen subspace $S_x = A_0 \wedge A_2 \wedge A_3$ with respect to the isotropic cone C_x. This can be done in the same way as we did for a lightlike hypersurface $V \subset (M,c)$ whose conformal structure c is of Lorentzian signature, $c = CO(n-1,1)$. The family of straight lines A_0A_4 associated with a hypersurface V is an isotropic congruence (see [AG 99b]) each ray of which bears two singular points F_1' and F_2'. To complete our specialization of moving frames, we choose a frame whose vertex A_4 coincides with the harmonic pole H' of the point A_0 with respect to singular points F_1' and F_2' (see Figure 3). Since the singular points are defined in a fourth-order differential neighborhood of a point $x \in V$, the point A_4 is defined also in this neighborhood.

Thus we arrive at the following result.

Theorem 6 *If $D \neq 0$, a canonical frame bundle on a lightlike hypersurface $V \subset (M,c), c = CO(2,2)$ is defined by elements of a fourth-order differential neighborhood of a point $x \in V$.*

Figure 3

References

[A 96] Akivis, M. A., *On the real theory of four-dimensional conformal structures*. J. Geom. Phys. **21** (1996), 55–80.

[AG 96] Akivis, M.A., and V. V. Goldberg, *Conformal differential geometry and its generalizations*, John Wiley & Sons, New York, 1996, xiii+383 pp.

[AG 98] Akivis, M.A., and V. V. Goldberg, *The Darboux mapping of canal hypersurfaces*, Beiträge Algebra Geom. **39** (1998), no. 2, 396–411.

[AG 99a] Akivis, M.A., and V. V. Goldberg, *The geometry of lightlike hypersurfaces on manifolds endowed with a conformal structure of Lorentzian signature*, Proceedings of Intern. Conf. on Differential Geom. and Appl. (Brno, Czech Republic, 1998) (to appear).

[AG 99b] Akivis, M.A., and V. V. Goldberg, *Lightlike hypersurfaces on manifolds endowed with a conformal structure of Lorentzian signature* (submitted).

[Ak 87] Akutagawa, K., *On space-like hypersurfaces with constant mean curvature in the de Sitter space*, Math. Z. **196** (1987), 13–19.

[Be 96] Bejancu, A., *Geometry of degenerate hypersurfaces*, Arab J. Math. Sci. **2** (1996), no. 1, 1–38. (1987), 337–361.

[BCGGG 91] Bryant, R. L., S. S. Chern, R. B. Gardner, H. L. Goldsmith, and P. A. Griffiths, *Exterior differential systems*, Springer-Verlag, New York, 1991, vii+475 pp.

[DB 91] Duggal, K. L. and A. Bejancu, A., *Degenerate hypersurfaces of semi-Riemannian manifolds*, Bull. Inst. Politehn. Din —aşi, Secţ. I **37** (1991), no. 1–4, 13–22.

[DB 96] Duggal, K. L. and A. Bejancu, A., *Lightlike submanifolds of semi-Riemannian manifolds and Applications*, Kluwer Acad. Publ., 1996, vii+300 pp.

[HE73] Hawking, S. W., and G. F. R. Ellis, *The large scale structure of spacetime*, Cambridge Univ. Press, London, 1973, xi+391 pp.

[Ku 87] Kupeli, D. N., *Degenerate submanifolds in semi-Riemannian geometry*, Geom. Dedicata **23** (1987), 337–361.

[Ku 96] Kupeli, D. N., *Singular semi-Riemannian geometry*, Kluwer Academic Publishers Group, Dordrecht, 1996, x+177 pp.

[O 62] Ostianu, N.M: *On a canonization of a moving frame of an embedded manifold*, (Russian) Rev. Roumaine Math. Pures Appl. **7** (1962), no. 2, 231–240.

[PR 86] Penrose, R., and W. Rindler, *Spinors and space-time*, vol. 2: *Spinor and twistor methods in space-time geometry*, Cambridge Univ. Press, Cambridge, 1986, x+501 pp.

[R 87] Ramanathan, J., *Complete space-like hypersurfaces of constant mean curvature in the de Sitter space*, Indiana Univ. Math. J. **36** (1987), 349–359.

[Ro 71] Rosca, R., *Sur les hypersurfaces isotropes de défault* 1 *incluses dans une variété lorentzienne*, C. R. Acad. Sci. Paris Sér. A, **272**, 393–396.

[Y 75] Yau, S. T., *Submanifolds with constant mean curvature II*, Amer. Math. J. **97** (1975), 349–359.

[Z 96] Zheng, Y., *Space-like hypersurfaces with constant scalar curvature in the de Sitter spaces*, Differential Geom. Appl. **6** (1996), 51–54.

Authors' addresses:

M. A. Akivis
Department of Mathematics
Jerusalem College of Technology
– Mahon Lev, P. O. B. 16031
Jerusalem 91160, Israel

E-mail address: akivis@avoda.jct.ac.il

V. V. Goldberg
Department of Mathematics
New Jersey Institute of Technology
University Heights
Newark, NJ 07102, U.S.A.

E-mail address: vlgold@numerics.njit.edu